Simplified Algorithm for Implementing an ABCD Ray Matrix Wave-Optics Propagator

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Outline

- Introduction & Motivation
  - Ray Matrices
  - Siegman Decomposition Algorithm
- Modifications to the Siegman ABCD Decomposition Algorithm
  - Simplification by Removing One Step
  - Addressing Degeneracies and Details
- Comparison of ABCD and Sequential Wave-Optics Propagation
- Conclusions

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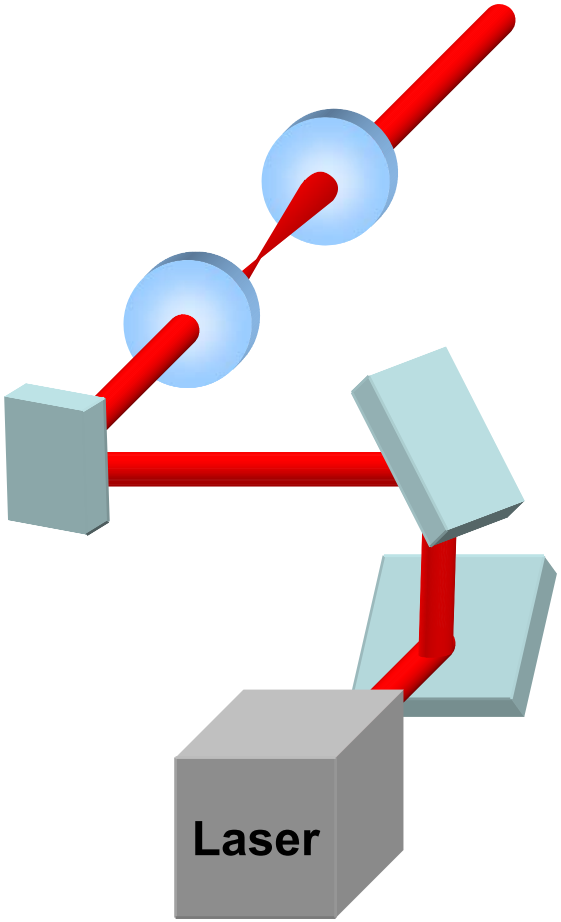
Introduction & Motivation

- Model propagation of a beam through a complex system of simple optics in as few steps as possible.
- We developed a technique for using ray matrices to include image rotation and reflection image inversion in wave-optics modeling.
- Here we introduce a technique to prescribe a wave-optics propagation using a ray matrix.



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Ray Matrix Formalism

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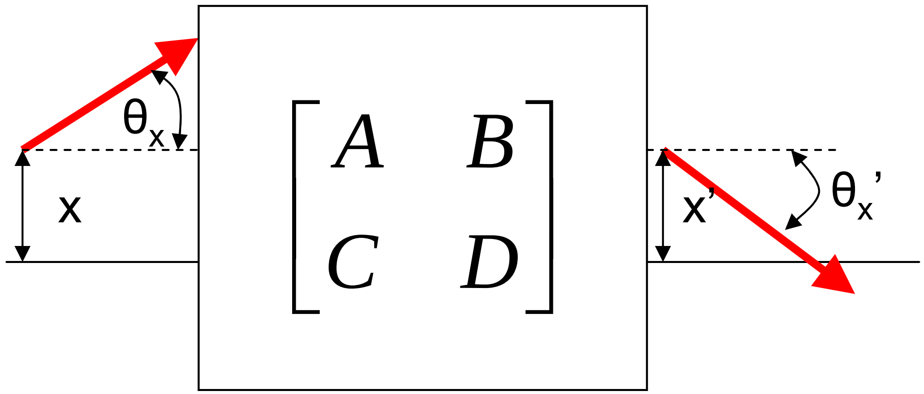
Introduction - Ray Matrices

- The most common ray matrix formalism is 2x2
  - a.k.a. ABCD matrix
- It describes how a ray height, x, and angle, θ_x, changes through a system.



$$\begin{bmatrix} x' \\ \theta_x' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ \theta_x \end{bmatrix}$$

$$x' = Ax + B\theta_x$$

$$\theta_x' = Cx + D\theta_x$$

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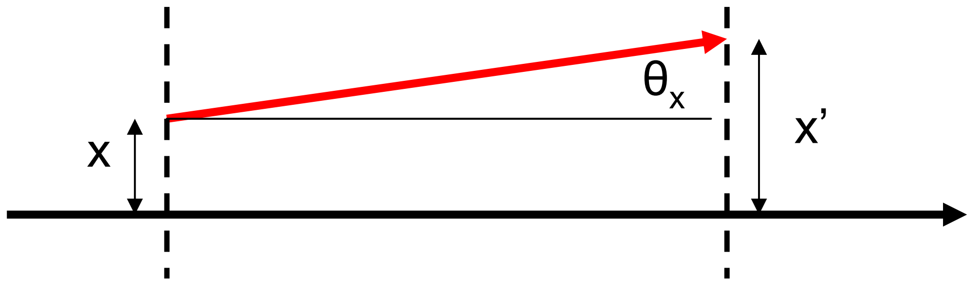
2x2 Ray Matrix Examples

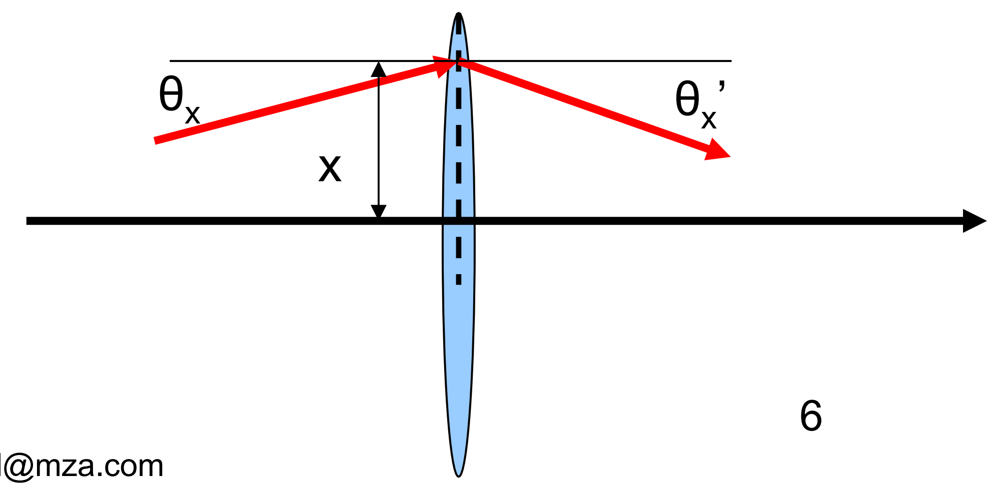
Propagation



$$x' = x + \theta_x \cdot L$$

$$\begin{bmatrix} x' \\ \theta_x' \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ \theta_x \end{bmatrix}$$

Lens



$$\theta_x' = \theta_x - x / f$$

$$\begin{bmatrix} x' \\ \theta_x' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} x \\ \theta_x \end{bmatrix}$$

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Example ABCD Matrices

Matrix Type	Form	Variables
Propagation	$\begin{bmatrix} 1 & L/n \\ 0 & 1 \end{bmatrix}$	L = physical length n = refractive index
Lens	$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$	f = effective focal length
Curved Mirror (normal incidence)	$\begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix}$	R = effective radius of curvature
Curved Dielectric Interface (normal incidence)	$\begin{bmatrix} 1 & 0 \\ -(n_2 - n_1)/R & 1 \end{bmatrix}$	n_1 = starting refractive index n_2 = ending refractive index R = effective radius of curvature

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3x3 and 4x4 Formalisms

- Siegman's *Lasers* book describes two other formalisms: 3x3 and 4x4
- The 3x3 formalism added the capability for tilt addition and off-axis elements.
- The 4x4 formalism included two-axis operations like axis inversion and image rotation.

3 x 3

$$\begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ \theta_x \\ 1 \end{bmatrix}$$

E = Offset
F = Added Tilt

4 x 4

$$\begin{bmatrix} A_x & B_x & & \\ C_x & D_x & & \\ & & A_y & B_y \\ & & C_y & D_y \end{bmatrix}\begin{bmatrix} x \\ \theta_x \\ y \\ \theta_y \end{bmatrix}$$

Dual-Axis ABCD

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5x5 Formalism

- We use a 5x5 ray matrix formalism as a combination of the 2x2, 3x3, and 4x4.
  - Previously introduced by Paxton and Latham
- Allows modeling of effects not in wave-optics.
  - Image Rotation
  - Reflection Image Inversion

Image Rotation

$$\begin{bmatrix} A_x & B_x & 0 & 0 & E_x \\ C_x & D_x & 0 & 0 & F_x \\ 0 & 0 & A_y & B_y & E_y \\ 0 & 0 & C_y & D_y & F_y \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ \theta_x \\ y \\ \theta_y \\ 1 \end{bmatrix}$$

Dual-Axis ABCD

Tilt and Offset

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Ray Matrix Wave-Optics Propagation Introduction

- Introduced a way of applying effects captured by a 5x5 ray matrix model with wave-optics.
  – Image Inversion
  – Image Rotation
- This relied on a parallel sequential wave-optics model and integration of these effects at the end.
- We complete the integration technique here by showing how the residual dual-axis ABCD matrices embedded in a 5x5 ray matrix can be used to specify a wave-optics propagation.

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ABCD Ray Matrix Wave-Optics Propagator

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Implementation Options

- Siegman combined the ABCD terms directly in the Huygens integral.
  - Less intuitive
  - Cannot obviously be built from simple components
- He then also introduced a way of decomposing any ABCD propagation into 5 individual steps.

$$U_2(x_2, y_2) = \frac{\exp(jkL)}{j\lambda B} \cdot$$

$$\int\int_{-\infty}^{\infty} U_1(x_1, y_1) \exp\left[\frac{jk}{2B} \left(\begin{array}{l} A\left(x_1^{\ 2} + y_1^{\ 2}\right) - \\ 2\left(x_1 x_2 + y_1 y_2\right) + \\ D\left(x_2^{\ 2} + y_2^{\ 2}\right) \end{array} \right) \right] dx_1 dy_1$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \begin{bmatrix} M_2 & 0 \\ 0 & 1/M_2 \end{bmatrix} \cdot$$

$$\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & 1/M_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix}$$

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Siegman Decomposition Algorithm

- Choose magnifications M_1 & M_2 ($M=M_1*M_2$)
- Calculate the effective propagation length and the focal lengths.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \Rightarrow$$

$$L_{eq} = \frac{L}{M_1{}^2} = \frac{B}{M}$$

$$f_1 = \frac{B}{M - A}$$

$$f_2 = \frac{B}{1/M - D}$$

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Modifications to the Siegman Decomposition Algorithm

- We found that one of the magnification terms was unnecessary (M_1=1.0).
- We modified Siegman's algorithm to better address two important situations:
  - image planes and
  - focal planes.
- We worked on how add diffraction into choosing magnification.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} M & 0 \\ 0 & 1/M \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} M & 0 \\ -1/Mf & 1/M \end{bmatrix}$$

$$D_2 = AD_1 + 2\eta \frac{L\lambda}{D_1}$$

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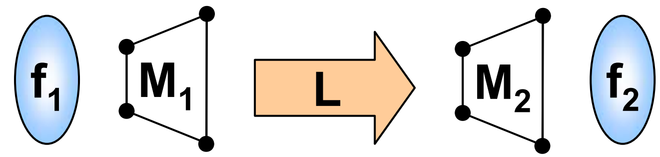
Eliminating a Magnification Term

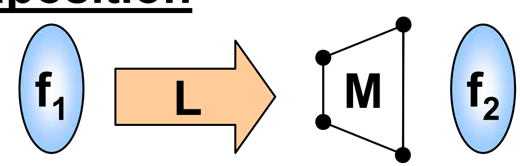
- We determined that one of the two magnification terms that Siegman put into his decomposition was unnecessary.
  - There were five steps (f_1,M_1,L,M_2,f_2)and four inputs (ABCD).

Original Decomposition



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \begin{bmatrix} M_2 & 0 \\ 0 & 1/M_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & 1/M_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix}$$

New Decomposition



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} M & 0 \\ 0 & 1/M \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix}$$

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Image Plane: B=0

- This case is an image plane.
- There is no propagation involved here, but there is
  - curvature and
  - magnification.

$$\begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix}\begin{bmatrix} M & 0 \\ 0 & 1/M \end{bmatrix} = \begin{bmatrix} M & 0 \\ -M/f_2 & 1/M \end{bmatrix}$$

Siegman

$$L_{eq} = \frac{B}{M} = 0$$

$$f_1 = \frac{B}{M-A} = 0$$

$$f_2 = \frac{B}{1/M-D} = 0$$

Our Algorithm

$$L_{eq} = 0$$

$$\mathrm{C = \text{-}1/Mf_2}$$

$$f_2 = \frac{-1}{MC}$$

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Automated Magnification Determination: Problems with the Focal Plane

$$\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} = \begin{bmatrix} 0 & f \\ -1/f & 1 \end{bmatrix}$$

- We were trying to automate the selection of the magnification by setting it equal to the A term of the ABCD matrix.
  - This minimizes the mesh requirements
- In doing so, we found that the decomposition algorithm was problematic at a focal plane.

Siegman, M=A

$$M = A = 0 \rightarrow$$

$$L_{eq} = \frac{f}{M} = \infty$$

$$f_1 = \frac{f}{M - A} = \frac{\mathrm{f}}{0}$$

$$f_2 = \frac{f}{\infty - 1} = 0$$

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Propagation to a Focus: A=0

$$\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} = \begin{bmatrix} 0 & f \\ -1/f & 1 \end{bmatrix}$$

- For a collimated beam going to a focus, this ray envelope diameter is zero.
- To handle this case, we force the user to specify the magnification.
- We also give the user guidance on how to choose magnification when there is substantial diffraction…

Siegman, M=A

$$M = A = 0 \rightarrow$$

$$L_{eq} = \frac{f}{M} = \infty$$

$$f_1 = \frac{f}{M - A} = \frac{\mathrm{f}}{0}$$

$$f_2 = \frac{f}{\infty - 1} = 0$$

Siegman, M=1

$$M = 1 \rightarrow$$

$$L_{eq} = \frac{B}{M} = \mathrm{f}$$

$$f_1 = \frac{B}{M - A} = \mathrm{f}$$

$$f_2 = \frac{B}{1/M - D} = 0$$

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Choosing Magnification while Considering Diffraction

- We propose here to add a diffraction term to the magnification to avoid the case of small M.
- We added a tuning parameter, η, which is the number of effective diffraction limited diameters.

$$D_2 = AD_1 + 2\eta \frac{L\lambda}{D_1}$$

$$M = \frac{D_2}{D_1} = A + 2\eta \frac{L\lambda}{D_1^{\ 2}}$$

$$= A + \frac{\eta}{2} \frac{1}{N_f}$$

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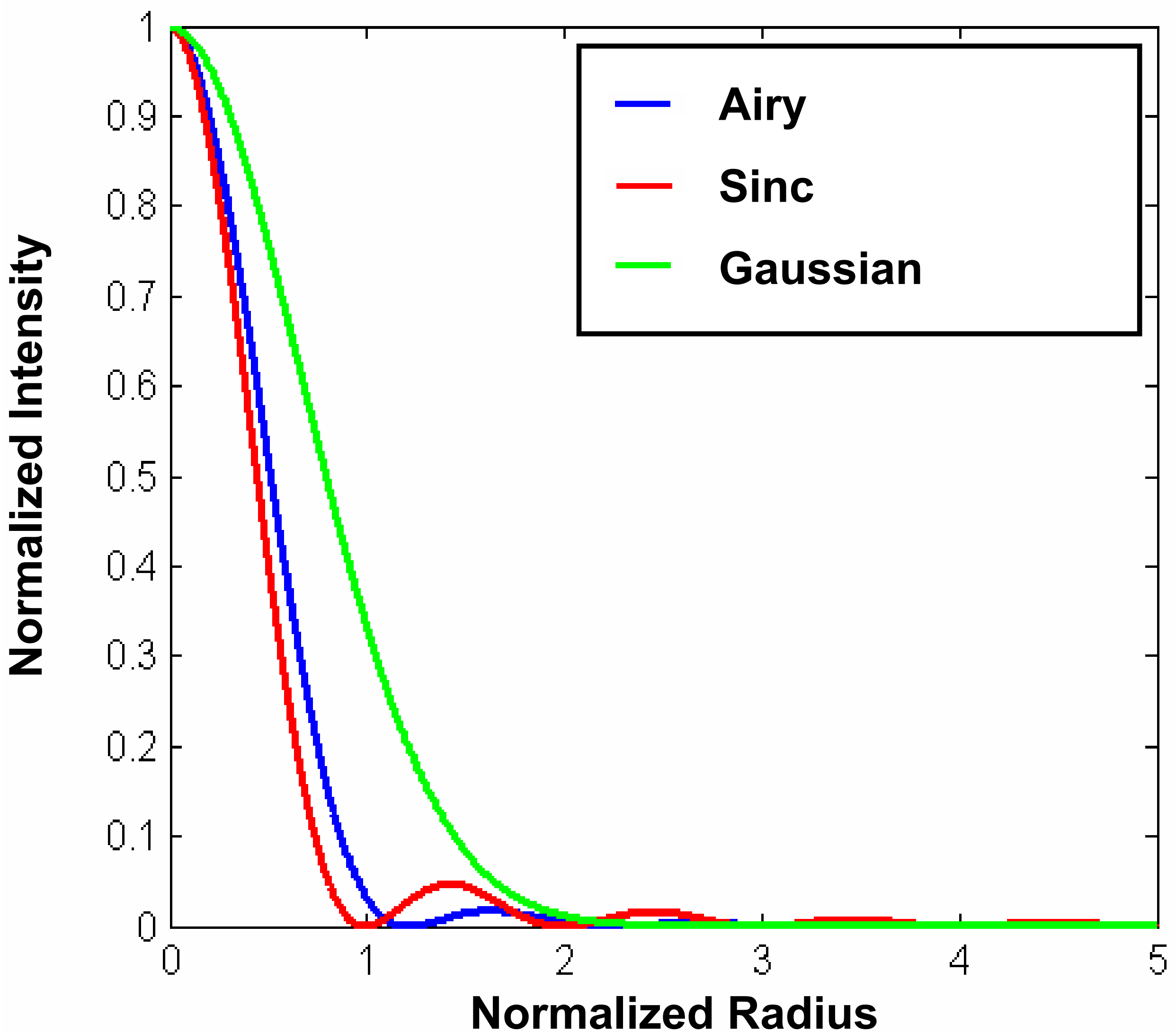
Common Diffraction Patterns



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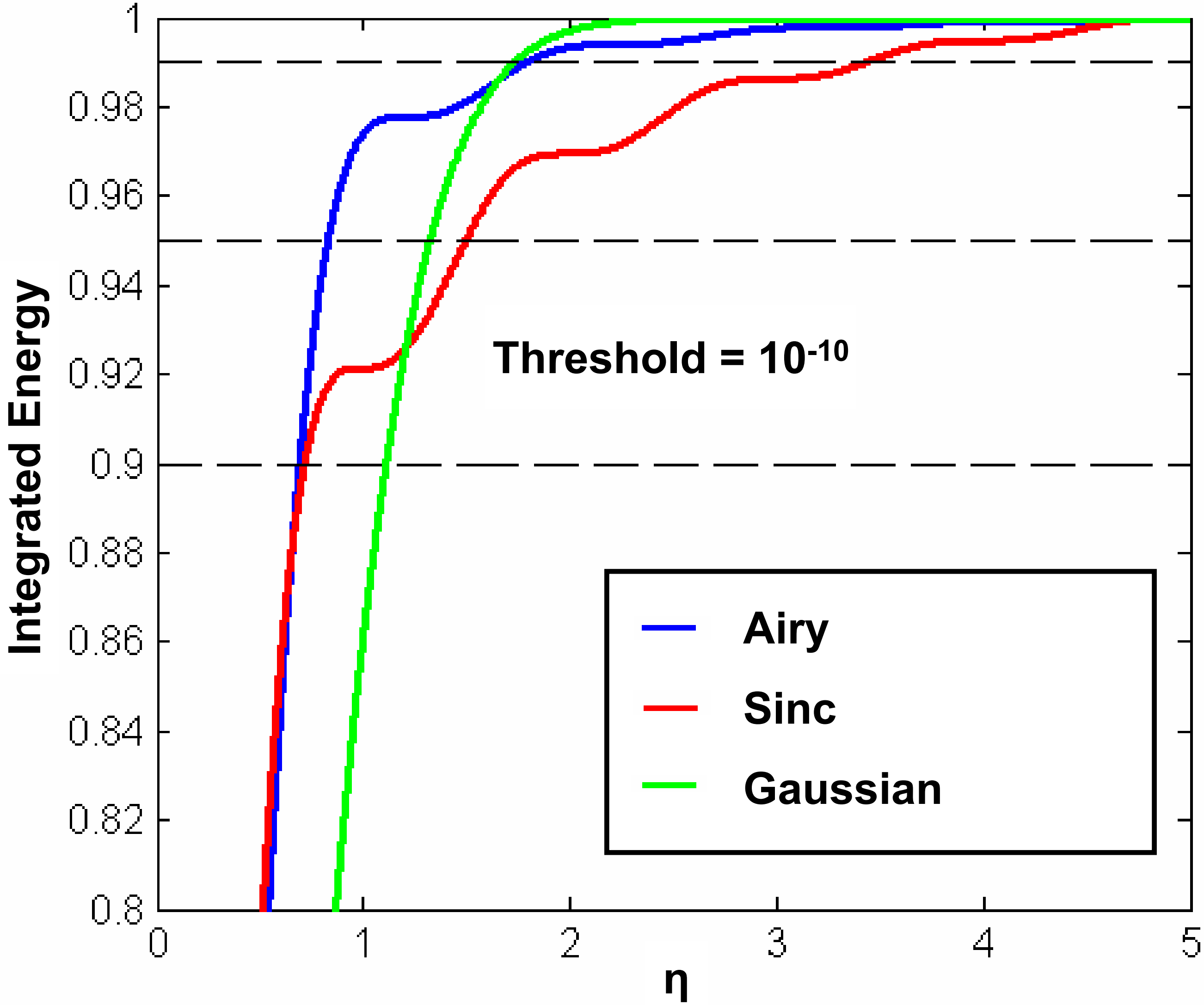
Integrated Energy



We concluded that η=5 is sufficient to capture more than 99% of the 1D integrated energy.

Modified Decomposition Algorithm

- If at an image plane (B=0)
  - ❑M=A (possible need for interpolation)
  - ❑Apply focus
- Else
  - ❑Specify M, considering diffraction if necessary
  - ❑Calculate and apply the effective propagation length and the focal lengths.

$$\begin{bmatrix} M - \frac{LM}{f_1} & LM \\ \frac{LM}{f_1 f_2} - \frac{1}{Mf_1} - \frac{M}{f_2} & \frac{1}{\mathrm{M}} - \frac{\mathrm{LM}}{\mathrm{f}_2} \end{bmatrix}$$

$$\text{for } \mathrm{M}_1 = 1.0$$

$$L_{eq} = \frac{L}{M_1^{\ 2}} = \frac{B}{M}$$

$$f_1 = \frac{B}{M - A}$$

$$f_2 = \frac{B}{1/M - D}$$

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Wave-Optics Implementation Details

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Implementing Negative Magnification

- After going through a focus, the magnification is negated.
- We implement negative magnification by inverting the field in one or both axes.
  - We consider the dual axis ray matrix propagation using the 5x5 ray matrix formalism.

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Dual Axis Implementation

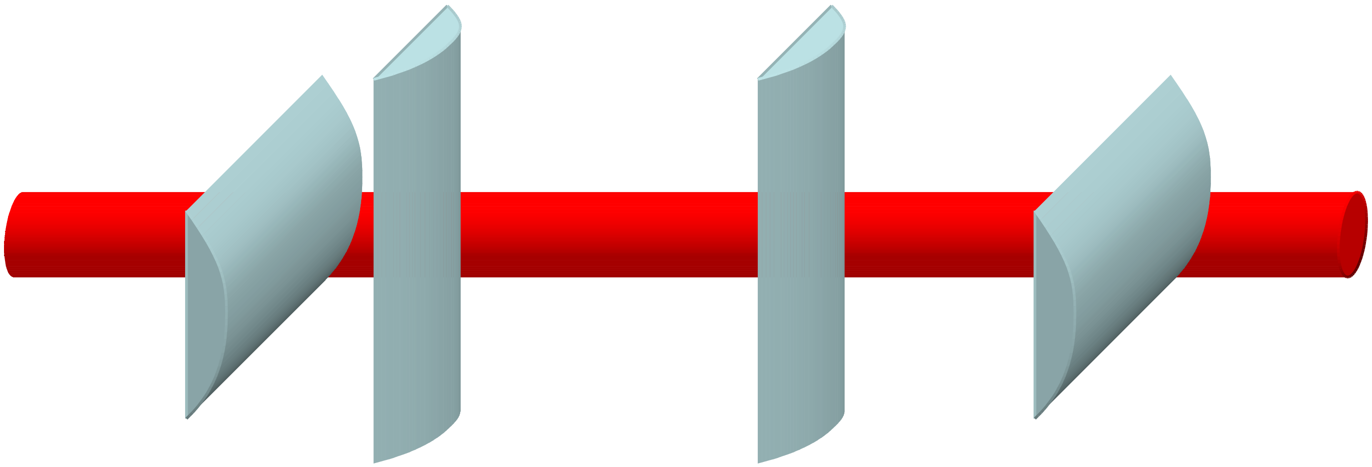
- Cylindrical telescopes along the axes are handled by dividing the convolution kernel into separate parts for the two axes.

$$U_2 = P \cdot F^{-1}\left(H \cdot F\left(U_1\right)\right)$$

$$H = \exp\left[-j\pi\lambda\left(z_x f_x^{\,2} + z_y f_y^{\,2}\right)\right]$$

$$F(x) = \text{Fourier Transform of } x$$

$$P = \text{Phase Factor}$$



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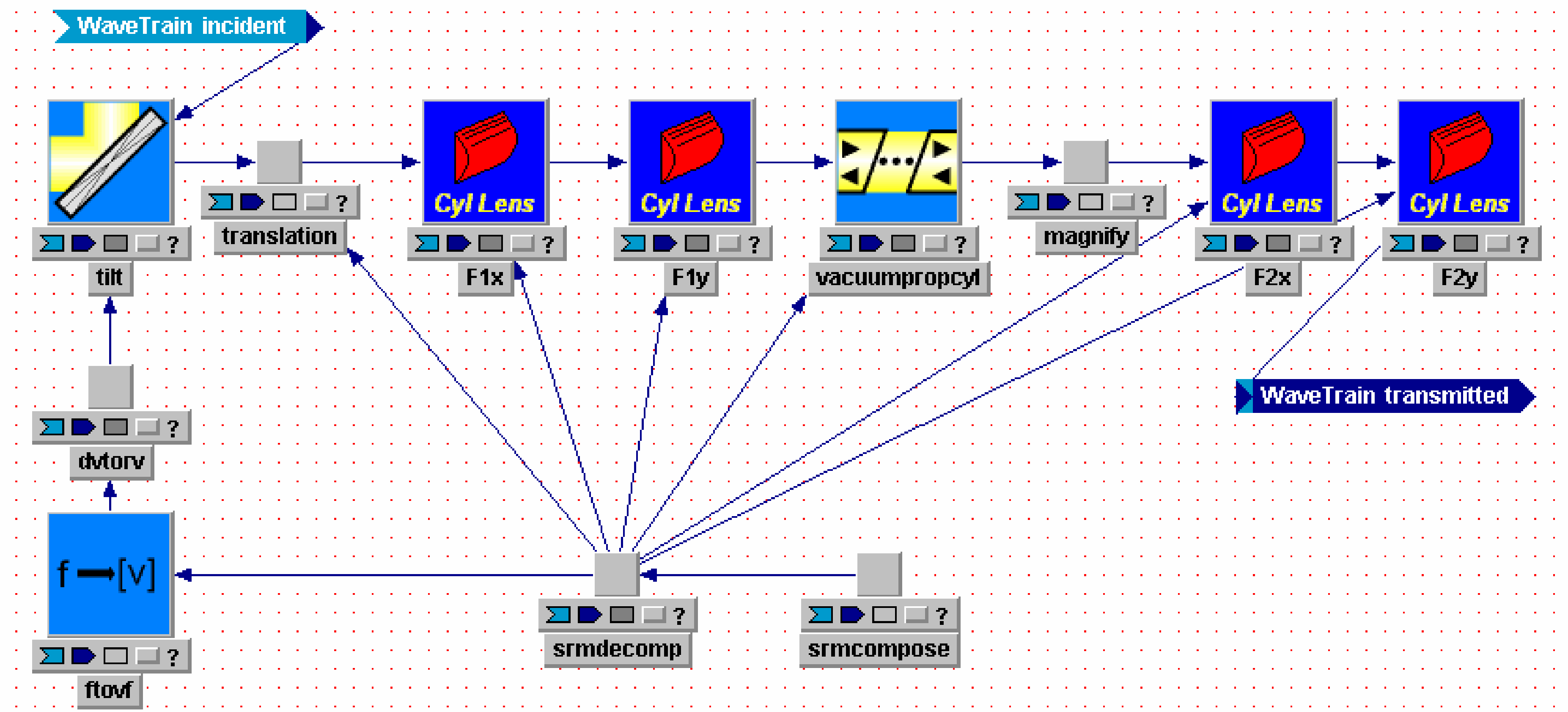
WaveTrain Implementation



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Example: ABCD Propagator

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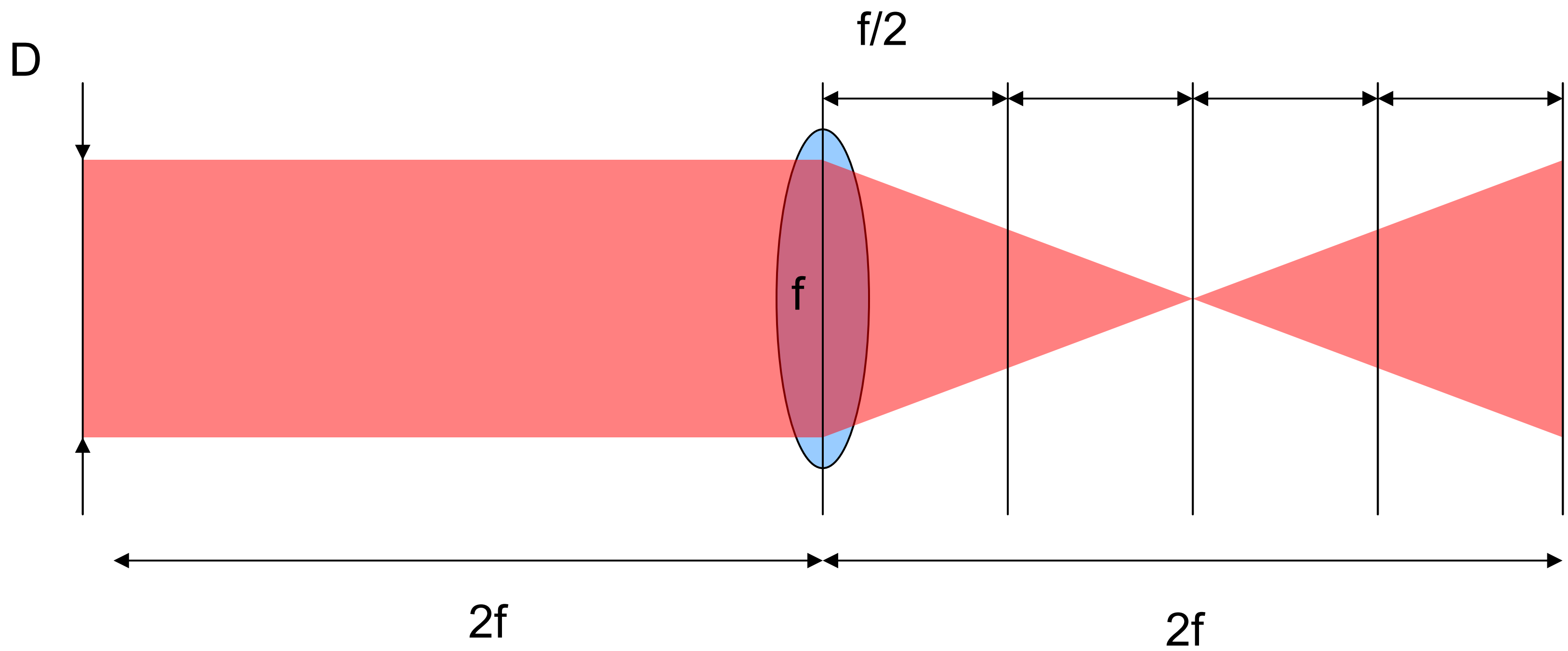
Example System



- Compared sequential and ABCD propagation fields

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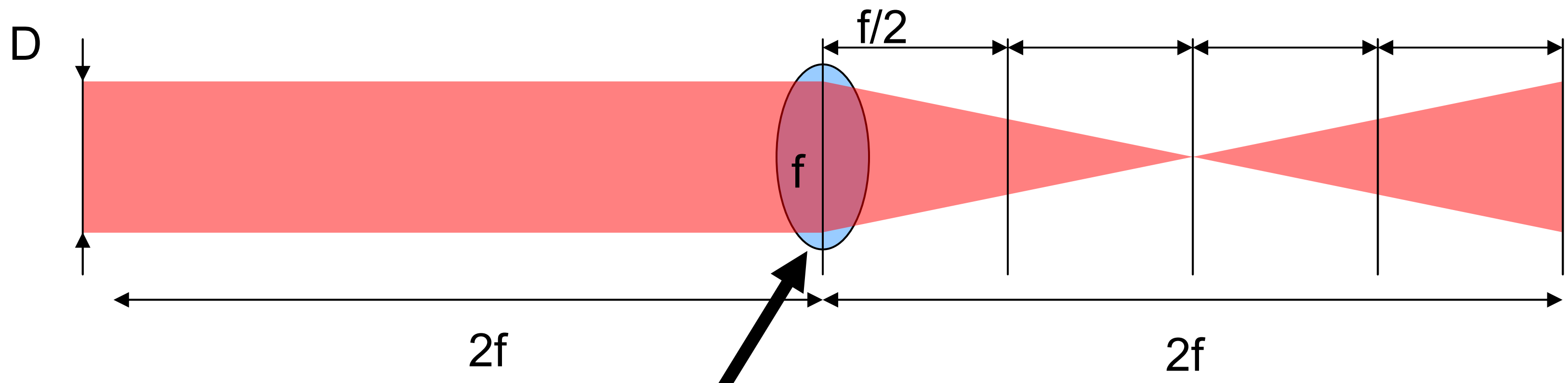
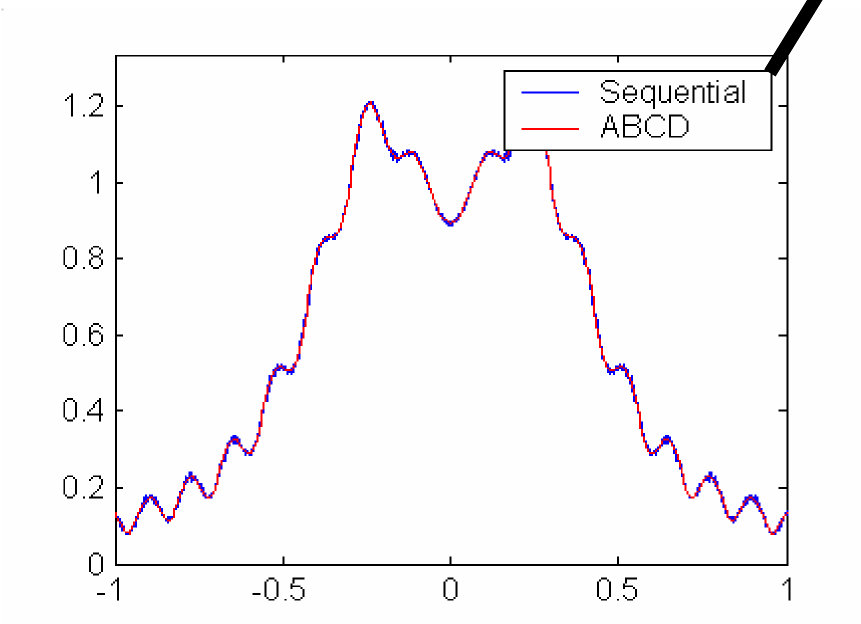
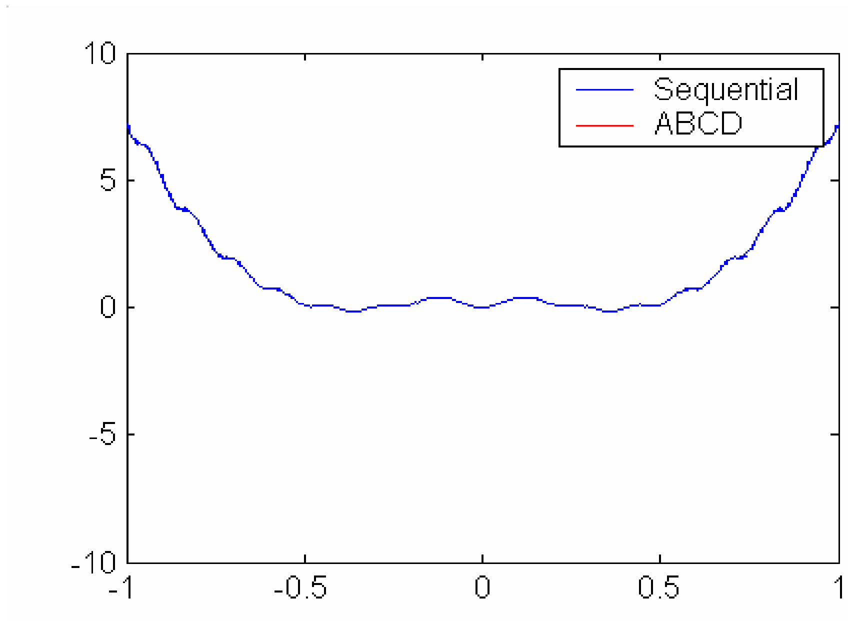
Field before the Lens





Field Magnitude



Field Phase

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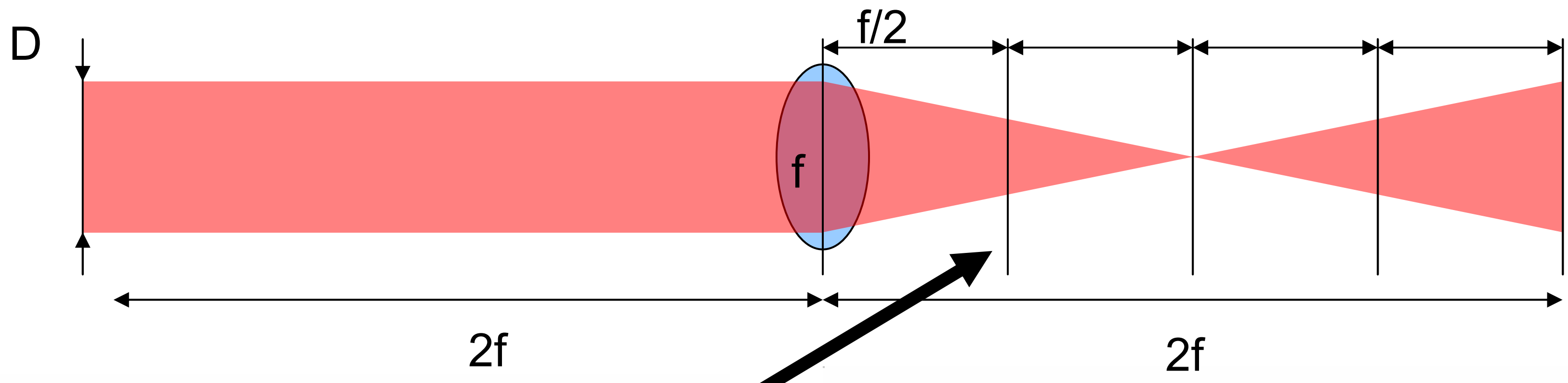
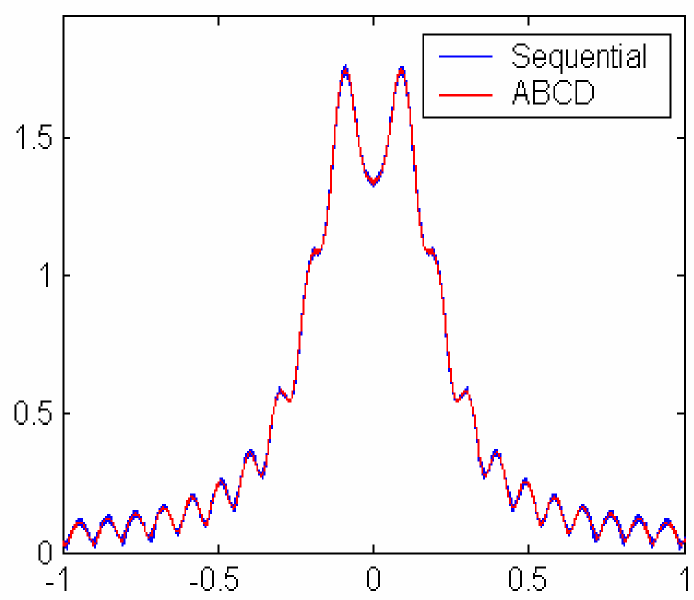
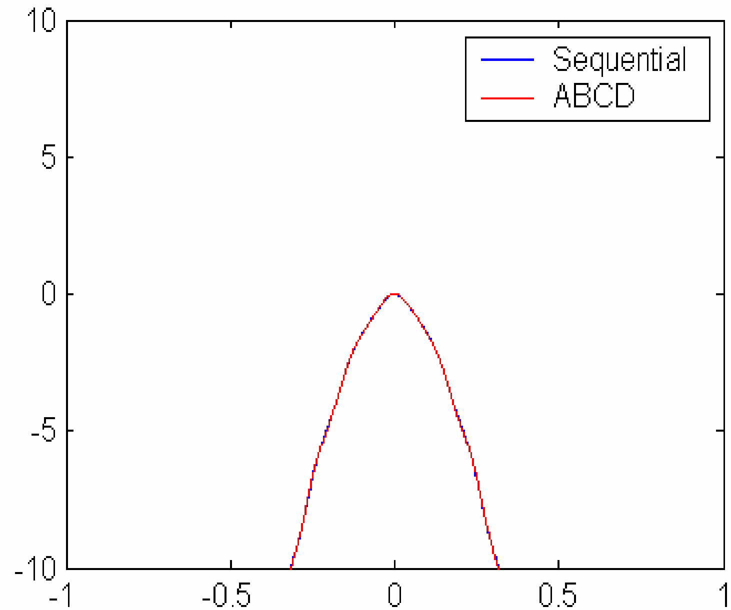
Field After Lens by Distance f/2





Field Magnitude



Field Phase

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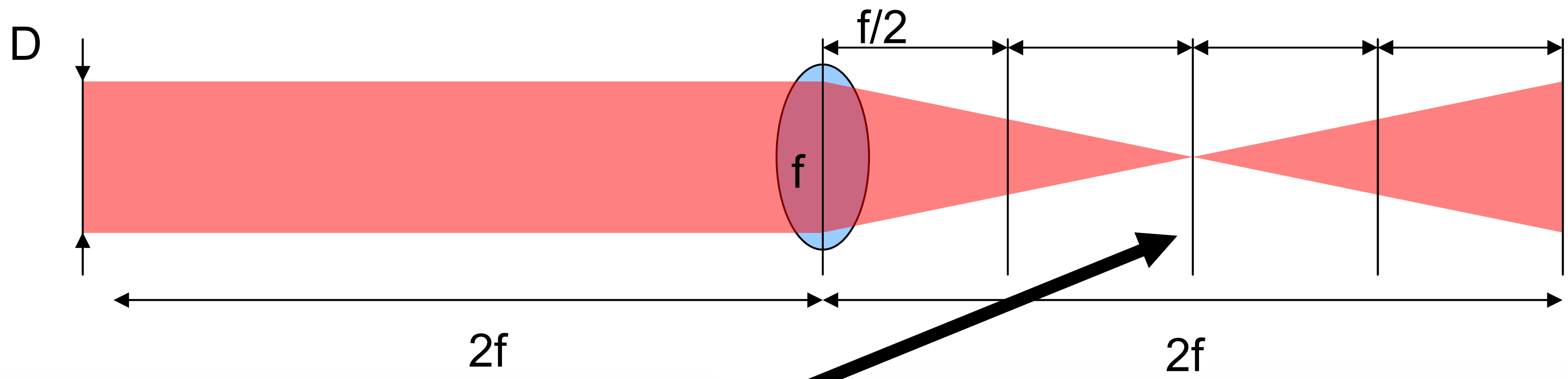
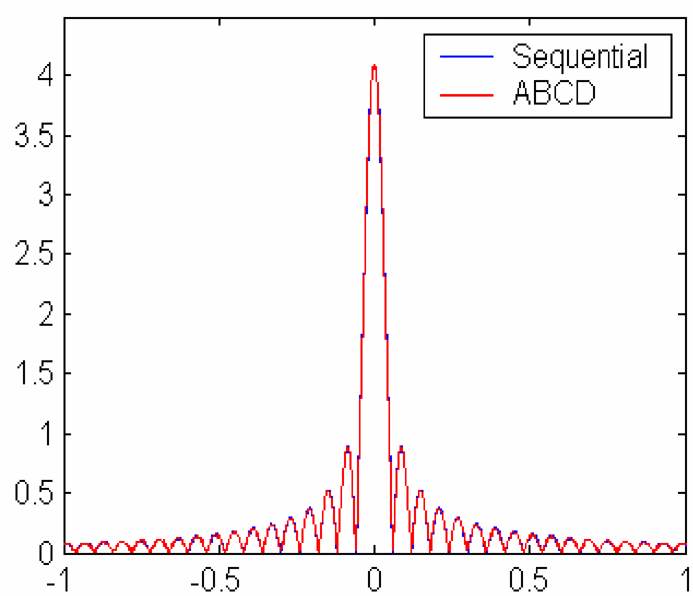
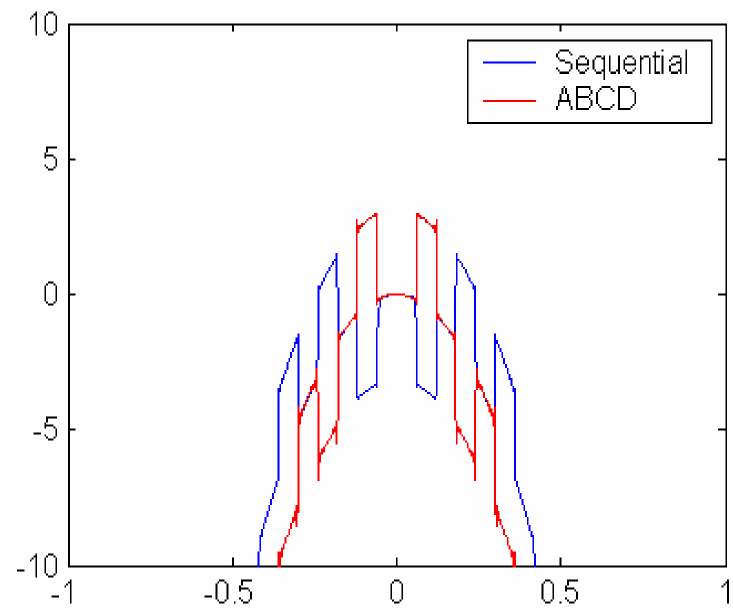
Field After Lens by Distance f





Field Magnitude



Field Phase

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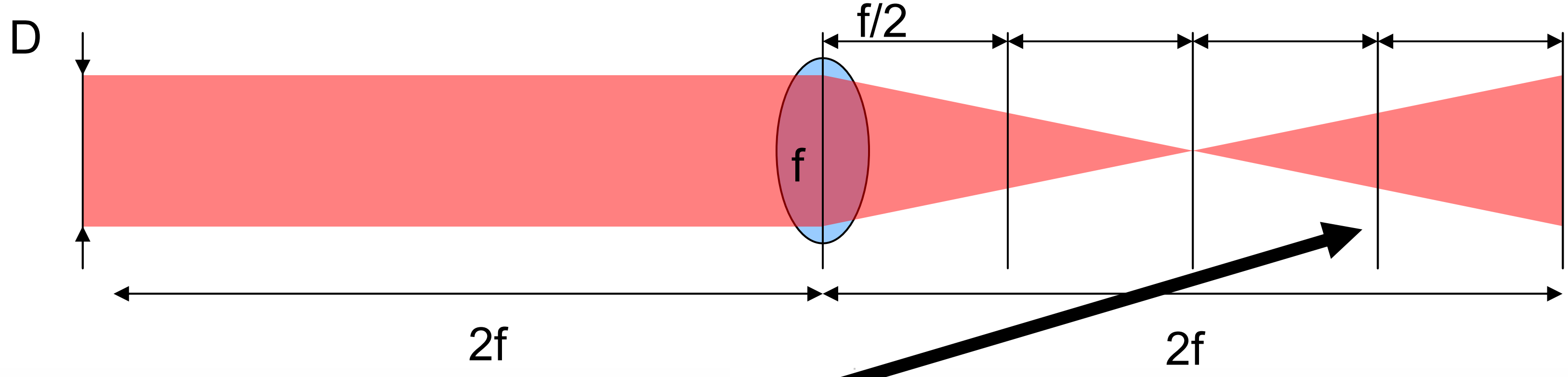
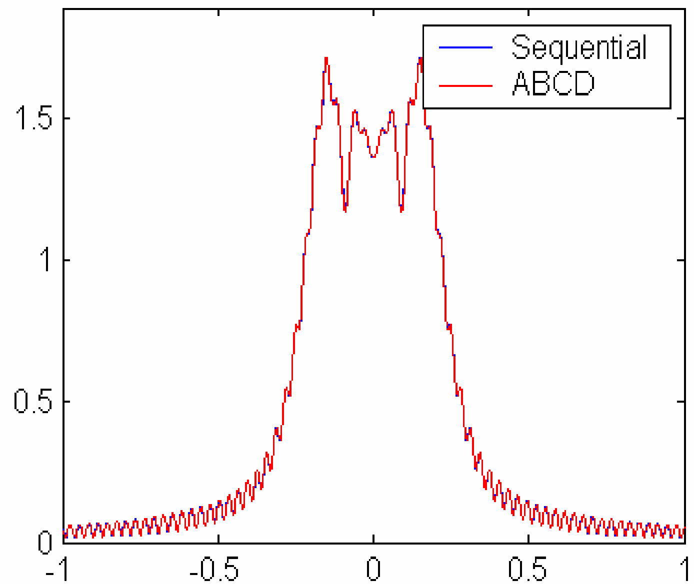
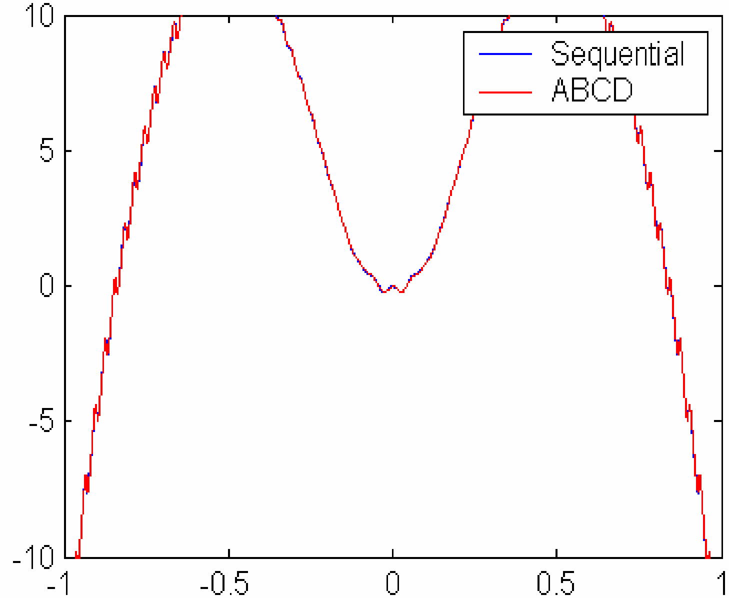
Field After Lens by Distance 3f/2





Field Magnitude



Field Phase

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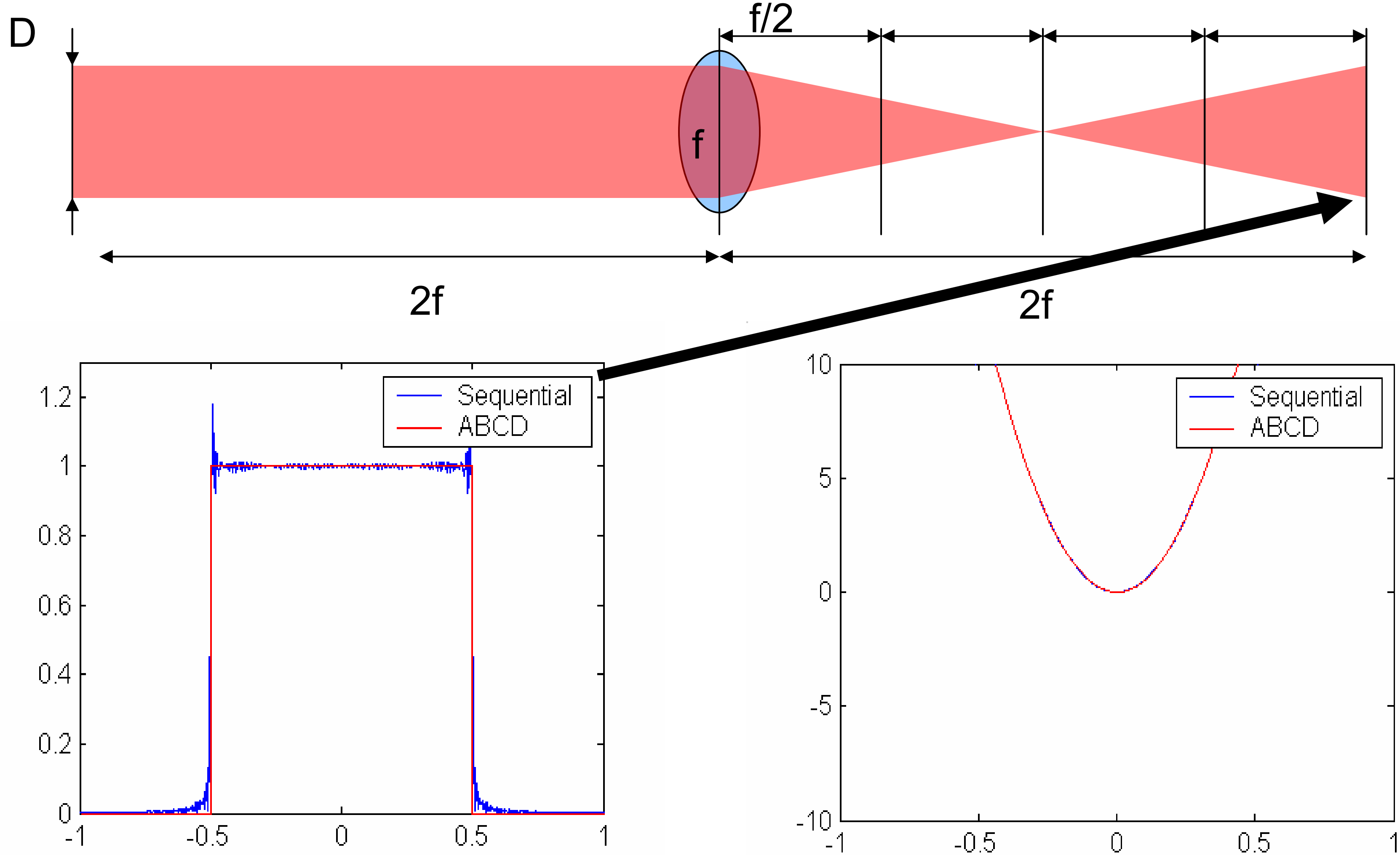
Field After Lens by Distance 2f



Field Magnitude **Field Phase**

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ABCD Ray Matrix Fourier Propagation Conclusions

- We have modified Siegman's ABCD decomposition algorithm to
  - remove one of the magnifications and
  - include several special cases such as
    - Image planes
    - Propagation to a focus
- This enables complex systems comprised of simple optical elements to be modeled in 4 steps (one Fourier propagation).

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Questions?

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